SURFACE AREA COMPUTATION IN ANCIENT MEXICO: DOCUMENTARY EVIDENCE OF ACOLHUA-AZTEC PROTO-GEOMETRY

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Abstract: *Two codices from the Valley of Mexico painted ca. AD 1540 show hieroglyphic drawings of household agricultural parcels. Decipherment of these indigenous manuscripts has revealed important information about Mesoamerican science before the Conquest. Reported here are documentary evidence from the Codex Vergara of surface area computation and results of a search for geometric algorithms that may have been employed by Acolhua-Aztec surveyors. Unappreciated for five centuries, the Acolhua-Aztec art of cadastral recordkeeping attests to their development of mathematical thought.*

The Aztec Empire encountered by Hernan Cortes in AD 1519 consisted of an alliance between three Nahuatl-speaking polities in the Valley of Mexico: the Mexica (Tenochca) occupying their island capital of Tenochtitlan in the center of Lake Texcoco, the Tepanec of Tlacopan on the lake's western shore, and the Acolhua of Tetzcoco on the mainland to the east. From this core the Triple Alliance conquered territory and peoples stretching from the Caribbean to Pacific shores and south to Guatemala. Access to natural resources and tribute-paying vassals guided the conquests of the Triple Alliance (Berdan et al., 1996; Carrasco, 1999).

Recordkeeping of various sorts and at varying territorial levels was required to manage the far-flung empire and complex imperial relations between Triple Alliance members. Imperial-level records have survived both in text and pictorial manuscripts detailing specific conquests during the reigns of particular kings and lists of tribute assessments. Crucial also to the functioning of the Triple Alliance Empire were local level records, such as population censuses and land tenure cadastrals from which tribute in goods and services could be assessed. Several documents of this genre are known from the early Contact Period. Among them are two pictorial codices from Tepetlaoztoc (Glass, 1975). This town in the eastern Valley of Mexico once was a city-state within the Acolhua Kingdom of Tetzcoco. The two codices are books drawn about AD 1540 in native hieroglyphics. They record household censuses and household land cadasters of 16 small villages and hamlets in the jurisdiction of Tepetlaoztoc. Known as the Codice de Santa Maria Asuncion and the Codex Vergara, these two pictorial documents received scant scholarly attention until recently, even though they had been assessed in the l9th century by the national libraries of Mexico and France, respectively (Codice de Santa Maria Asuncion, Ms; Codex Vergara, Ms). Lack of attention was due, at least in part, to the fact that no one had been able to decode large sections of the cadastral information. That changed two decades ago when Harvey and Williams deciphered the numerical portions of the Codice de Santa Maria Asuncion (Harvey and Williams, 1980). Their work showed that Acolhua-Aztec surveyors of the Contact Period carefully measured and recorded linear dimensions of agricultural fields. And, more importantly, these surveyors also determined surface area of land parcels, which they recorded in a unique numerical notation system.

That the Acolhua surveyors conceptualized surface area as an attribute of land parcels is significant in the history of Mesoamerican mathematical science. It suggests the existence of an indigenous geometry. This, in turn, leads us to query how indigenous area values were derived. Building on the Harvey and Williams study, we address the question of area mensuration through an analysis of one of the Tepetlaoztoc manuscripts, the Codex Vergara. Our preliminary results suggest that by the time of European Contact, the Acolhua were practitioners of a proto-geometry. They apparently conceptualized areal attributes of geometric forms such as rectangles and triangles, and evidence strongly suggests they calculated surface areas using several algorithms requiring the functional equivalents of addition, subtraction, multiplication and division. To our knowledge, this is the first instance of documentary, empirical evidence of mathematical computation applied to surveying and cadastral recordkeeping in Contact Period Central Mexico.

To develop our argument, we first discuss the structure of the manuscript data and the metrologic systems employed by the surveyors. Next we describe our Codex Vergara database, followed by examples illustrating what we hypothesize to be Acolhua surface area algorithms.

STRUCTURE AND METROLOGY OF THE LAND RECORDS

The Codex Vergara consists of population censuses and landholding cadasters of households which resided in five villages in the jurisdiction of Tepetlaoztoc (Calatlaxoxiuhco, Topotitla, Teocaltitla, Patlachiuhca, and Texcalticpac). Each locality has three sequential sections of recorded data. The first section shows the demographic composition of each household in the locality. At the end of household population sections, a second section follows showing cadastral drawings of the land parcels belonging to the enumerated households. The third data section for each locality shows another cadastral register of the same landholdings, but in a different format. Why indigenous surveyors recorded the same agricultural fields twice becomes evident by comparing the two cadasters. Each is distinctly different from the other (Figure 1).

Acolhua surveyors had two systems of land measurement. In Figure 1, in [A], the land parcels 1-4 of household head Alonso Yhuiltemoc are drawn to show field shape and linear perimeter dimensions. In this type of register, called *milcocoli*, lines equal one standard-length linear unit, the *tlalquahuitl*. Groups of 5 lines read “5 standard-lengths,” and a dot equals 20 standard-lengths. The same fields 1-4 of Alonso are shown in [B], but a different numerical notation system is used in this cadaster, called *tlahuelmatli*. The stylized rectangles representing land parcels function as information frames. Values ascribed to the numerical symbols depend upon their position/place in the frame. Lines in the upper-right tab (Place 1), as in Field 1, are units of 1. In contrast, lines on the bottom margin (Place 2), as in Field 4, or in the center, equal 20 units (1 x 20). Dots, found only in the center of the frame (Place 3), equal 400 units (20 x 20), as in Fields 1, 2 and 3. When the number is less than 400 units, as in Field 4, a corncob glyph appears at the top margin of the frame, effectively indicating "zero" in Place 3. These numbers in the *tlahuelmatli* cadaster refer to the surficial area in square *tlalquahuitl* of the land parcels shown in the *milcocoli* cadaster. For example, the *milcocoli* cadaster [A] for Field 4 shows a rectangular field 20 standard-lengths long and 10 standard-lengths wide. The area calculated from these dimensions is thus 200 square *tlalquahuitl*. The *tlahuelmatli* cadaster [B] records this value by 10 lines x 20 [= 200] on the bottom margin of the information frame. Non-numerical symbols in the center of the drawings in both cadasters refer to the soil type of the field (Codex Vergara folio 9 verso and 17 recto).

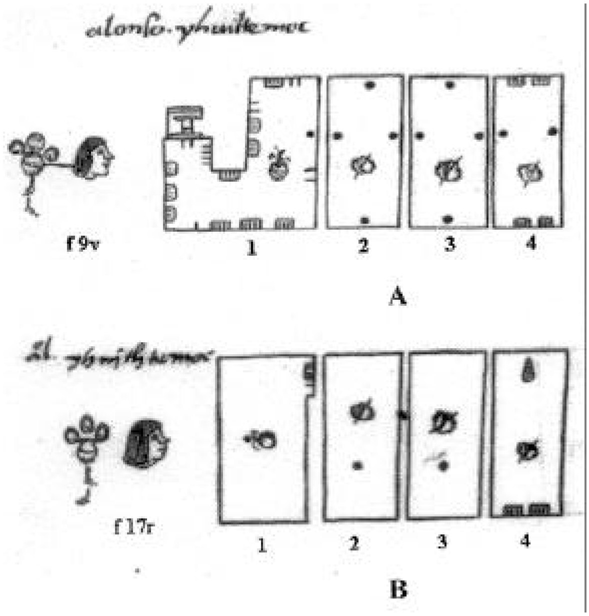

Figure 1

In the first cadaster, the surveyor/scribe sketched the shape of each field but without precise linear or angular scale. This register, labeled *milcocoli* in the codex, also shows field perimeter measurements. The length of each side of a field is indicated using a combination of lines and dots. One line equals one standard-length measure called the *tlalquahuitl*. Four lines with a line above reads "5 standard-length measures," and a solid dot reads "20 standard-length measures." At the end of the sequence of lines and dots, glyphs of a hand, heart, or arrow sometimes appear. These glyphs represent distances shorter than the standard length, which were to be added to the *tlalquahuitl* length. The *milcocoli* drawings also record soil type by a glyph in the center of each parcel. A detailed explication of numerical and soil glyphs is found in Williams and Harvey (1997, pp. 30-37).

The second cadaster, called *tlahuelmatli* in the codex, shows the same fields as the *milcocoli* register, but with significant differences. One is that the fields all have the same size and shape, that of a rectangle often with a tab in the upper right corner. Thus, the *tlahuelmatli* cadaster does not convey information about the field shape; rather, the rectangle serves as an information frame. Secondly, the numerical notation system is different. Number signs are composed of lines and dots exclusively, and they appear only in specific places in the frame. Lines appear in the tab, on the bottom line, and in the center. Solid dots occur only in the middle of the frame, along with the soil-type glyph. The meaning of the numbers was finally deciphered by Harvey and Williams (1980) when they discovered that a rudimentary form of positional (place) notation is applied in this cadaster. Lines in the tab (Place 1) are units of 1, whereas lines on the bottom line (Place 2) or in the center of the field (Place 3) are units of 20 (1 x 20). Dots, which occur only in the center of the field, are also multiplied by 20, so that one dot is a unit of 400 (20 x 20). Place 1 records values from 1-19 units; Place 2 records values from 20-399; Place 3 records units of 400 or greater. When the value of the total number is less than 400, a corncob glyph is drawn near the top margin of the frame to draw attention to the fact that the number is less than 400, effectively meaning "zero" in Place 3.

The numerical values recorded in this second cadaster denote the surface area of each field in square *tlalquahuitl.* This conclusion results by calculating the surface areas of fields using the measurements in the *milcocoli* cadaster, and then comparing the calculated areas to the indigenous recorded numbers in the *tlahuelmatli* cadaster (Figure 2).

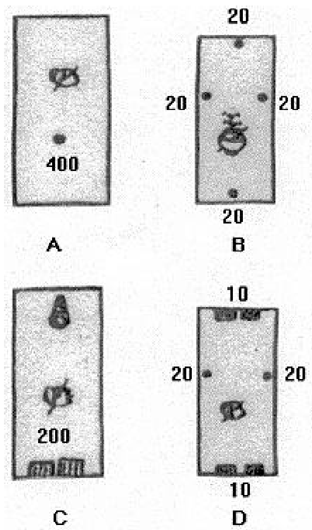

Figure 2

In Figure 2 numbers shown on fields [A] and [C] from the *tlahuelmatli* cadaster read 400 and 200, respectively. From perimeter dimensions of the same fields shown in the *milcocoli* cadaster [B, D], area may be computed by multiplying length times width, equaling 400 (20 x 20) and 200 (20 x 10) square *tlalquahuitl*, respectively. Exact correspondence obtains between recorded and calculated areas for 120 or 122 fields in which all sides or opposite sides are equal. Congruence between calculated field areas and indigenous numerical values provides evidence that the *tlahuelmatli* cadaster records field areas in square *tlalquahuitl* (Codex Vergara, folio 16 recto [A, C], folio 8 verso [B,D]).)

VERGARA DATABASE AND ANALYSIS

For this study[1] we calculated areas of 282 four-sided, quadrangular fields in the Codex Vergara to compare with the indigenous area record. We assumed that field side-angles equaled either 90°, or the maximum angle possible given the quadrangle side-lengths, an assumption consistent with contemporary and historical agricultural field patterns. Also, our computations ignored the smaller-than-standard distances of hearts, hands, and arrows. The exact correspondence between our areas and the indigenous ones, where these smaller measures were indicated in the *milcocoli* cadaster, implies that the Acolhua surveyors also ignored the smaller length measures when determining area. The tallies of correspondence between our calculations and the indigenous ones do not include the irregularly-shaped fields with more than four sides. However, we present here examples to illustrate how the Acolhua surveyors probably deconstructed these irregular shapes into rectangles and triangles. Finally, we should note that our calculations use the base-10 decimal system. Our research in progress addresses the fact that Acolhua surveyors and mathematicians would have used the pan-Mesoamerican, base-20 system.

For analytical purposes we classed the quadrangles into three groups: (1) those in which opposite sides are equal, (2) those in which only one pair of opposite sides is equal, and (3) those in which no opposite sides are equal. To identify potential algorithms, we hypothesized that for presumed squares and rectangles (quadrangles with equal opposite sides), surveyors used either the Length-times-Width Rule or the Surveyor's Rule, where area equals the product of the average lengths of opposite sides. In cases where these algorithms produced discrepancies, by trial-and-error we tried other methods of computation which might provide the best fit with the surveyors' area values.

Quadrangles with equal opposite sides

For fields in which opposite sides are equal, multiplying length times width results in field areas corresponding exactly to the indigenous record in 120 of 122 cases. This nearly perfect correspondence in so many examples suggests that the Acolhua used the Length-times-Width algorithm.

Quadrangles with one pair of equal opposite sides

For fields in which only one pair of opposite sides is equal (n = 49), we derived the Codex area figures most frequently by applying the Surveyor's Rule. Using this algorithm, we achieve exact correspondence in 19 of 36 fields. For the 17 fields in which there are not exact correspondences, we find an average of 6.5 and a median of 3.5 square *tlalquahuitl* difference between our values and the recorded ones.

Two other algorithms provide the best fit for the remaining 13 parcels in this quadrangle class. In three cases, area exactly equals the product of one equal side and the longest adjacent side. A converse rule (area equals the product of one equal side and the shortest adjacent side) gives exact correspondence in six instances. In four cases of application of these algorithms, discrepancies of 3 square *tlalquahuitl* obtain.

Quadrangles in which no opposite sides are equal

Quadrangles in which no opposites sides are equal (n = 111) show more variability in algorithms which may have been applied, and less congruence between our calculations and indigenous ones, probably because of greater irregularity in field shape. We achieved best fit using the following algorithms: Surveyor's Rule (n = 80); the product of the average of one pair of opposite sides and the longest adjacent side (n = 6); the product of the average of one pair of opposite sides and the shortest adjacent side (n = 6); the product of the longest length and the shortest width (n = 8); the product of the shortest length and the longest width (n = 5); the product of the shortest length and shortest width (n = 4); and, the product of the average length of adjacent (rather than opposite) sides (n = 2). In the application of these algorithms, we have 18 cases of exact correspondence contrasted with 93 discrepancies between our values and the indigenous ones.

Quadrangle Type and Applicable Algorithm	Number of Quad-rangles	Number of Exact Correspon-dences	Number of Discre-pancies	Average and Median Discrepancy in square *tlalquahuitl*
Opposite sides equal				
Length x width	122	120	2	7.0; median=na
One pair opposite sides equal				
Surveyor's Rule	36	19	17	6.3; median=3.5
Equal side x longest adjacent side	4	3	1	2.0; median=na
Equal side x shortest adjacent side	9	6	3	2.4; median=na
Sub-totals	49	28	21	5.5
No equal sides				
Surveyor's Rule	80	6	74	5.6; median=3
Average of one pair of opposite sides x longest adjacent side	6	1	5	7.5; median=3
Average of one pair of opposite sides x shortest adjacent side	6	5	1	6.0; median=na
Longest length x shortest width	8	4	4	3.5; median=2
Shortest length x longest width	5	1	4	5.5; median=5
Shortest length x shortest width	4	1	3	7.0; median=5
Product of average of adjacent sides	2	0	2	3.5; median=na
Sub-totals	111	18	93	5.6
Grand total	282	166	116	5.6

Table 1: Recorded land areas of quadrangles in the Codex Vergara *tlahuelmatli* cadasters compared with areas calculated from dimensions given in the *milcocoli* registers. Number of exact correspondences and discrepancies by algorithms applied to each quadrangle type.

Table 1 summarizes the number of exact correspondences, and number and magnitude of discrepancies for the ten algorithms which provide the best fit with recorded values in the total quadrangle database (n = 282). Exact match is achieved for approximately 60% of the recorded field areas. In the 40% where discrepancies occur, the average discrepancy does not exceed 7.5 square measures; medians and modes are about half that number. Thus, it seems reasonable to conclude that various permutations of the Length-times-Width Rule and the Surveyor's Rule served as indigenous algorithms to derive surface area values of quadrangular agricultural fields.

Areas of Triangles and Trapezoids

As mentioned, the *milcocoli* sketches of land parcels are not drawn to scale. As a result, some parcels appear to approximate rectangular shape, but when the recorded side-lengths are taken into account, the shapes are far from rectangular. In some rare instances, however, the scribes deliberately drew land parcels as triangles and trapezoids. These special cases provide instances from which to draw further insight into Acolhua geometry.

There are three triangle-shaped fields in the Codex. For one, the Acolhua area value in the *tlahuelmatli* register is 93 square *tlalquahuitl* (Figure 3). Whereas surveyors could measure height of triangles on the ground, we used the Pythagorean theorem and computed the area using the algorithm of 1/2 base times height. We hypothesized that the Acolhua approximated the given triangle as an isosceles one. Why isosceles? Because if we divide any isosceles triangle through the height into two equal triangles, these parts will fit into a rectangle whose sides are the height and half the base of the original isosceles triangle. Therefore, assuming an isosceles triangle with equal side lengths of 18 and a base of 11 gives a height of 17.1, which rounded to 17 results in an area of 93.5 square *tlalquahuitl*, almost perfect correspondence with the recorded value of 93.

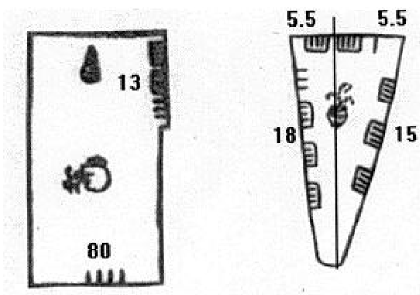

Figure 3

In Figure 3 the recorded area [A] of this triangle is 93 sq. *tlalquahuitl*. From the dimensions given in [B] and assuming an isosceles triangle with side lengths of 18, the triangle height is 17.1. Rounded down to 17, the triangle area is 93.5 sq. *tlalquahuitl* (17 x 5.5). Correspondence between recorded and computed areas strongly suggests that area was derived by Acolhua surveyors with an algorithm analogous to A = 1/2 base times height (Codex Vergara, folio18 verso [A**]**, folio 10 verso [B].

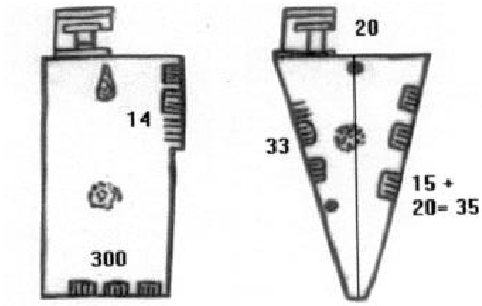

Figure 4

In a second example (Figure 4), the recorded area of the triangle in the *tlahuelmatli* register is 314 square *tlalquahuitl.* In the *milcocoli* drawing of this parcel, a scribe error seems quite apparent. The right leg dimension, shown as 15, lacks a dot so as to read 35, thus approximating the length of the left leg, which is 33. As in the previous case, assuming an isosceles triangle with side-lengths of 33, then the height of the triangle is 31.4. The height of 31.4 times 1/2 the base yields an area of 314, exactly as the indigenous record. In both of these cases correspondence between our values and the indigenous ones strongly suggests that the Acolhua used the equivalent of the well-known algorithm for triangle area.

In Figure 4, the recorded area [A] of this triangle is 314 square *tlalquahuitl*. The dimensions given in [B] of the right leg undoubtedly lack a dot symbol for twenty. Assuming an isosceles triangle with side lengths of 33, the height of the triangle is 31.4 and the area is 314. Exact correspondence between recorded and computed areas (with the recording error corrected) strongly suggests indigenous calculation of area by an algorithm analogous to Area = 1/2 base times height (Codex Vergara, folio 53 recto [A], folio 46 recto [B].

For trapezoidal-shaped fields, the Acolhua surveyors apparently visualized area computation as in Western geometry, where area equals the sum of the areas of a rectangle and two residual triangles. For example, Figure 5 shows a trapezoid whose parallel widths are 15 and 4 standard measures and whose lengths are 33 standard measures. By the Pythagorean Rule the actual height of the two right triangles is 32.5 and each base is 5.5 *tlalquahuitl*. By rounding down the height and base, the resultant area of the two triangles is 160 (32 x 5) square *tlalquahuitl*. Added to the area of the rectangle (4 x 32) gives a total of 288 square *tlalquahuitl*, exactly corresponding to the Codex recorded area.

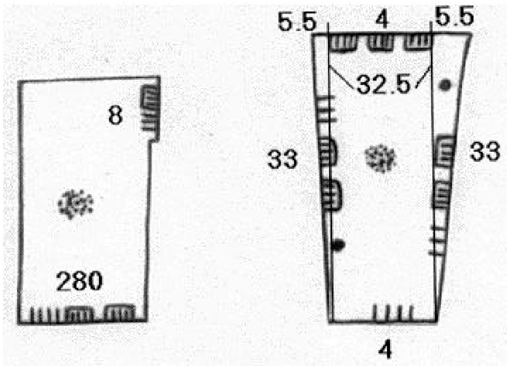

Figure 5

In Figure 5 the recorded area [A] of this trapezoid reads 288 square *tlalquahuitl*. Given the dimensions in [B], by the Pythagorean Rule the actual height of the two right triangles is 32.5, and each base is 5.5 *tlalquahuitl*. By rounding down the height and base, the resultant area of the two triangles is 160 (32 x 5), which when added to the area of the rectangle (4 x 32) gives a total of 288 square *tlalquahuitl*, exactly corresponding to the recorded area (Codex Vergara, folio 23 verso [A], folio 25 verso [B].

A second trapezoid drawn in the *milcocoli* register (Figure 6) shows parallel widths of 6 and 23 *tlalquahuitl*, and two lengths of 52 and 49 *tlalquahuitl*. The recorded area is 727 square *tlalquahuitl*. To approximate a trapezoid we average the two sides 52 and 49 to obtain 50.5. Because rounding up gives the same error as rounding down, therefore let us consider equal sides of length 51. For the two triangles with base 8.5, their height becomes 50.28 which we round down to 50. With a height of 50, then the area of the two triangles is 425 square *tlalquahuitl* (50 x 8.5), to which is added the rectangular area of 300 square *tlalquahuitl* (50 x 6) for a total area of 725 square *tlalquahuitl*. Our computed area is only two measures less than the recorded one. The calculation procedure utilized rounding, and the results suggest that Acolhua surveyors likewise may have employed some kind of rounding.

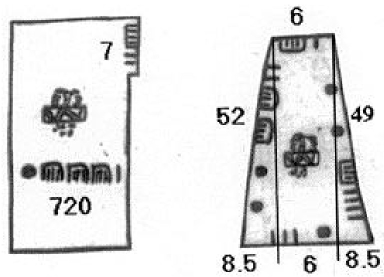

Figure 6

In Figure 6 the recorded area [A] of this trapezoid is 727 square *tlalquahuitl*. Assuming that the hypotenuse of each right triangle is 51 and the resultant height of 50.28 is rounded down to 50, then the area of the two triangles is 425 square *tlalquahuitl* (50 x 8.5). Adding the rectangular area of 300 square *tlalquahuitl* (50 x 6) gives a total area of 725 square *tlalquahuitl*, two units less than the recorded area. (Codex Vergara folio 32 recto [A] and folio 30 recto [B]).

Areas of Irregular Polygons

Our preliminary analysis of irregularly-shaped polygons suggests that they were conceptualized as combinations of rectangles and triangles. In some cases, the area of the largest rectangle was determined, and from it was subtracted the area which extended beyond the actual field boundary. Thus, as shown in Figure 7, area may be calculated by multiplying the length times width of the largest rectangle (31 x 7) and subtracting the area of the smaller rectangle (3 x 3) which falls outside of the actual field boundary. The computed area is 208 square *tlalquahuitl,* whereas the recorded area is 206 square *tlalquahuitl*. This small discrepancy between recorded and computed areas suggests that the Acolhua surveyor's were following a similar procedure.

In Figure 7 the recorded area [A] of this irregular polygon is 206 square *tlalquahuitl*. The area may be computed from the *milcocoli* dimensions [B] by multiplying length times width of the largest rectangle (31 x 7=217) and subtracting the area of the smaller rectangle (3 x 3) which falls outside of the actual field boundary, resulting in a total of 208 square *tlalquahuitl*. The two square unit discrepancy between recorded and computed areas suggests that Acolhua surveyors quite likely employed a similar method (Codex Vergara folio 49 verso[A] and folio 42 verso[B]).

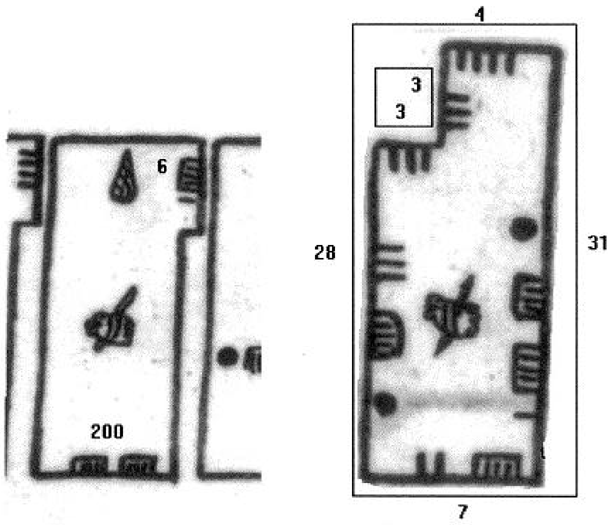

Figure 7

Figure 8 shows another example of an irregular polygon. Here the area may be computed by calculating the area of a rectangle formed by the longest field length times the shortest width (50 x 36=1800). Subtracting from that rectangle the area outside of the actual field boundary (20 x 25=500), and adding an additional triangular area of 4 x 40/2 results in a total area of 1380 square *tlalquahuitl*. The calculated area is only one square *tlalquahuitl* less than the recorded area of 1381 square *tlalquahuitl.* It should be noted that other area values for this land parcel may be computed. For example, constructing a large rectangle 40 x 36 (=1440) less the small rectangle (20 x 25=500) plus a triangle of 4 x 50/2 (=100) gives an area of 1040, much less than the recorded value. Conversely, given the large rectangle of 50 x 36 (1800), less the small rectangle of 25 x 20 (=500) plus a triangle of 4 x 50/2 (=100) yields an area of 1400 square *tlalquahuitl*, 19 more than the value recorded. These and other area values are possible because the land sketches are not drawn to scale and angular relationships between sides present infinite alternatives. Thus, a conjunction between recorded and calculated values again lends credence to our hypothesis that our computation method follows Acolhua procedure.

Derivation of areas of these irregularly-shaped fields required carrying out all four arithmetical operations-addition, subtraction, multiplication, and division. Interestingly, when confronted with large or irregular fields, both the Acolhua-Aztecs and the ancient Sumerians seem to have resolved areal measurement similarly (Nissen, Damerow and Englund 1993, pp. 68-69).

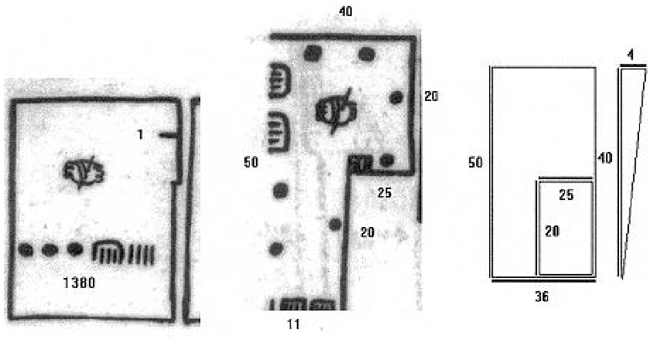

Figure 8

In Figure 8 the recorded area [A] of this irregular polygon is 1381 square *tlalquahuitl*. Given the perimeter dimensions in [B], an area of 1380 square *tlalquahuitl* results from computing the area of a large rectangle comprised by the longest side-length and the shortest width (50 x 36), subtracting the area of the smaller rectangle outside the field boundary (20 x 25), and adding the area of a triangle whose height is a sum of the shortest side dimensions (20 + 20) and whose base is the remaining distance of the top field margin (40-36=4; A=4 x 40/2). Constituent shapes are shown in [C]. The correspondence between recorded and computed area values provides evidence that the Acolhua surveyors deconstructed large and complex fields into rectangles and triangles. Codex Vergara, folio 56 recto [A] and folio 49 recto [B].

DISCUSSION AND CONCLUSIONS

Acolhua surveyors used two systems to establish the size of a field parcel, linear measurements and area measures. That the Acolhua employed the area concept is significant in itself. Failure to recognize this indigenous accomplishment has led to misinterpretation of numerical information on cadastral documents (Harvey and Williams 1986, pp. 251-254). A case in point is the mistaken interpretation of the Codex Mariano Jimenez from Otlazpan in the region of San Juan del Rio (Leander, 1967). It shows a vertical column of rectangular drawings with numerical notations indicating a standard width of 20 measures but with lengths increasing regularly in increments of 20 measures. Alongside the column of rectangles is a corresponding column of tribute items, among them cacao and coins, which also increase in quantity in a regular sequence. Rather than representing actual land parcels, these drawings quite likely are *tlahuelmatli*-like information frames recording land *area* measures and the tax rate correlated with land area.

The Acolhua derived areas of squares, rectangles, and triangles. Irregularly-shaped polygons apparently were seen as composites of these shapes. We are able to compute indigenous area values by application of various permutations of the Length-times-Width Rule, the Surveyor's Rule, and the algorithm for area of right triangles. While a conjunction of recorded and computed values does not constitute proof, it is highly likely that these were Acolhua geometric rules.

There is a striking resemblance between Acolhua-Aztec and archaic Sumerian land mensuration. The conclusions of Nissen, Damerow, and Englund (1993, pp. 55, 68) from their study of the archaic texts apply equally to Acolhua procedures. The Acolhua cadasters showing linear measurements were not drawn to scale, but distortions in the rendering of the field sketches had no real effect on area determination. The *milcocoli* records showing side-length dimensions served as work sheets from which areas could be computed and then recorded in the separate, *tlahuelmatli* cadasters. From the viewpoint of modern geometry, the various permutations of the Length-times-Width Rule and the Surveyor's Rule do not provide accurate area measurement, because angular relationships between sides are not considered. But for the Acolhua surveyors, these procedures constituted the area concept. They are perhaps best characterized as a proto-geometry, similar to – but three millennia later than – that of ancient Sumeria.

What might have prompted the development of the area concept among the Acolhua-Aztecs? One impetus may have come from local and imperial land redistribution policies. In this context, surface area provided a standard measure to quantitatively compare landholdings between households and across communities. Another and surely important impetus came from the taxation system, which assessed tribute according to the amount of land cultivated or held in usufruct. Of course, other polities in the Aztec world had similar land and taxation policies, so reevaluation of manuscript sources may reveal a more universal distribution of the areal concept within the Empire.

By the close of the l6th century, 90% of the indigenous population in Central Mexico had died. Simultaneously, their cadastral bookkeeping ended and apparently much indigenous mathematical knowledge was lost as well. Determining the level of mathematical knowledge present in Mesoamerica at the time of Contact has presented a challenge to researchers because of an apparent lack of ethnohistorical documentation from both native and Spanish sources. Thus, much of the discussion has focused on mathematics implicit in artifacts, architecture, archaeological site plans, astronomy, and calendrics (Vinette 1986). But, as the analysis of the Tepetlaoztoc codices show, there may be edited documents or unedited manuscripts in archives whose significance to ethnomathematics of Mesoamerican peoples has not been understood or appreciated. Ethnohistorical sources may yet provide significant research material for understanding mathematics of the peoples of the Triple Alliance and their neighbors.

NOTE

Earlier versions of this paper were presented in the symposiums *Ethnoscience and Ethnomathematics*: *The Evolution of Thought in the Last 500 Years*, Ubiratan D'Ambrosio and Paulus Gerdes, organizers and *Ciencia y Tecnologia en el Mexico Antiguo*, Teresa Rojas Rabiela and Jesus Galindo, organizers, at the XXII International Congress of the History of Science, Mexico City, July 8-15, 2001. A grant to B. Williams from the Mexican Academy of Science Distinguished Visiting Professors Program 2000-2001 and sponsorship by Dr. Teresa Rojas Rabiela, Professor of Ethnohistory, Center for Research and Graduate Study of Social Anthropology [CIESAS] facilitated investigation of Acolhua algorithms. Their support, and testing of the Surveyor's Rule by CIESAS workshop members, are gratefully acknowledged.

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